A Theory of Distributed Anonymous Mobile Robots —Formation and Agreement Problems

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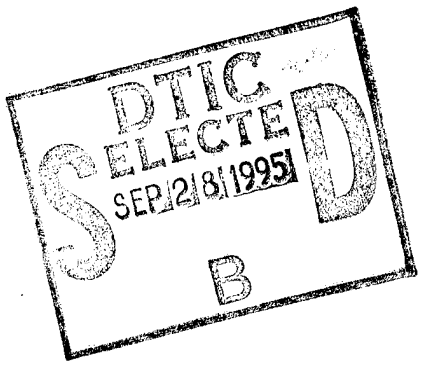

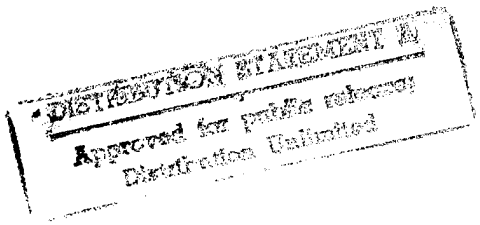

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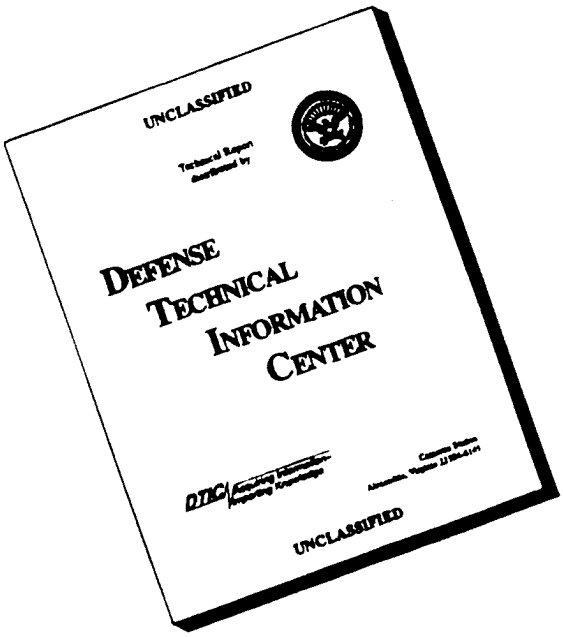

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A Theory of Distributed Anonymous Mobile Robots —Formation and Agreement Problems*

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Abstract A system consisting of multiple mobile robots in which the robots can see each other by their eye sensors but are not equipped with any communication system, can be viewed as a distributed system in which the components (i.e., robots) can "communicate" with each other only by means of their moves. We use this system to investigate, through a case study of a number of problems on the formation of geometric figures in the plane, the power and limitations of the distributed control method for mobile robots. In the distributed control method, each robot, at infinitely many unpredictable time instants, observes the positions of all the robots and moves to a new position determined by the given algorithm. The robots are anonymous in the sense that they all execute the same algorithm and they cannot be distinguished

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by their appearances. The robots are not necessarily synchronous, so they may not always observe their positions simultaneously. Furthermore, initially the robots do not have a common x-y coordinate system. The problems we discuss include (1) converging the robots to a single point, (2) moving the robots to a single point, (3) agreement on a single point, (4) agreement on the unit distance, (5) agreement on direction, and (6) leader election. We develop algorithms for solving some of these problems under various conditions. Some impossibility results are also presented.

1 Introduction

In the last several years, interest in the distributed control method for multiple mobile robots has increased considerably [1, 2, 7, 10]. The main idea of the method is to let each robot execute a simple algorithm and determine its movement adaptively based on the observed movement of other robots, so that the robots as a whole group will achieve the given goal. This approach has been shown to be very promising for the generation of certain patterns and collision avoidance. In the earlier works on distributed robot control, the main emphasis is on the development of heuristic algorithms for various problems and the presentation of simulation results, and in many cases, formal discussions on the correctness and performance of the algorithms are not given [1, 7].

A robot system in which the robots can communicate with each other by radio, such as a system of radio-controlled vehicles or spaceships, can be considered as a distributed system whose communication topology is a complete graph. Therefore, such systems can be analyzed using the standard techniques developed for distributed computing systems (although such analyses are by no means easy). In this paper, we consider a system consisting of multiple mobile robots in which the robots can see each other by their eye sensors, but they are not equipped with any communication system. The study reveals delicate interplay of a number of key concepts of distributed computing, such as synchrony and asynchrony, communication, termination detection, self-stabilization, anonymity of processors, and knowledge (in a casual sense).

A basic problem for such a robot system is to design an algorithm such that, if all the robots execute it individually, then the robots as a whole group will eventually form the given geometric figure, such as a circle and a line segment [4, 7, 8]. The main goal of this paper is to present some theoretical results related to this problem. The results presented here provide useful insights that will help us to answer certain fundamental questions, such as whether the given algorithm really solves the given problem and, for that matter, whether the given problem is solvable at all in a strict sense by a distributed algorithm. This work is a step toward the ultimate goal of determining exactly what class of problems are solvable in a distributed manner.

We assume that each robot is a mobile processor with infinite memory and an eye sensor, that repeatedly becomes *active* at unpredictable time instants. (At other times it is *inactive*.) Each time a robot becomes active, it observes the positions

of all the robots in terms of its own local x-y coordinate system, and moves to a new position determined by the given deterministic algorithm.[1] The algorithm is *oblivious* if the new position is determined only from the positions of the robots observed at that time instant. Otherwise, it is not oblivious, and the new position may depend also on the observations made in the past. To simplify the discussion, in this paper we assume that (1) the time it takes for a robot to move to its new position is negligibly small, and (2) a robot is a point (and hence two or more robots can occupy the same position simultaneously). These assumptions help us to bring out the fundamental issues of the problem, and still, many of the techniques and results we obtain for this simplified case seem to apply (with some modifications) to many realistic applications. (We plan to report on the case when the moves of a robot are not instantaneous in a future paper.) The robots are *synchronous* if they always become active simultaneously. Unless otherwise stated, we assume that the robots are not necessarily synchronous. We assume that initially, the robots do not have a common x-y coordinate system. So the local x-y coordinate systems of two robots may not agree on the location of the origin, the unit distance, or the direction of the positive x-axis. The robots are anonymous in the sense that (1) they do not know their identifiers, (2) they all use the same algorithm for determining the next position, and (3) they cannot be distinguished by their appearances. Since a robot observes other robots only at the moments when it becomes active, the third constraint implies that a robot that observes other robots at two time instants may not be able to tell which robot has moved to which position while it was inactive.

Two robots are said to be *clones* of each other if they have the same local x-y coordinate system and the same initial position, and they always become active simultaneously. Note that clones can never be separated. Throughout this paper, we assume that no clones exist in the system.

In order to give the reader a concrete image of the robot system, in Section 3 we review some of the known heuristic algorithms for converging the robots to two geometric figures, an approximation of a circle and a line segment. These algorithms are oblivious.

Then we start a formal discussion on the robot system. First, we consider a simple problem of converging the robots toward a single point. (The process of convergence need not terminate in finite steps.) Note that since the robots do not have a common x-y coordinate system, we cannot simply use an algorithm such as "move toward point $(0, 0)$". For this problem, we give two oblivious algorithms, and then discuss the subtlety of the problem by showing how certain minor changes in the algorithm affect the possibility of achieving the goal. We also consider a related problem of moving the robots to a single point *in finite steps*. Such a problem is called a *formation problem*, in contrast to a convergence problem. We show that this problem can be solved by a nonoblivious algorithm, but not solvable by any oblivious algorithm even for $n = 2$, where n is the number of robots. The corresponding convergence problem

[1] Nondeterministic algorithms that allow a robot to randomly select its next position from two or more candidates are out of the scope of this paper.

can be solved by an oblivious algorithm, as we stated above.

Second, we investigate the problem of having the robots agree on a common x-y coordinate system. (The term "agree" is defined formally in Section 2.) Clearly, such an agreement can greatly reduce the complexity of motion coordination algorithms. For example, convergence toward a point mentioned in the previous paragraph can easily be solved by moving all the robots toward point $(0, 0)$ of the common coordinate system. The problem consists of three subproblems, agreement on the origin, agreement on the unit distance, and agreement on the direction of the positive x-axis. We show that the first two agreement problems are solvable by nonoblivious algorithms, but the third problem is not solvable in general, even for $n = 2$. The last result shows that the robots cannot agree on a common x-y coordinate system in general.

Third, we consider the case in which the robots have a sense of direction, i.e., the direction of the positive x-axis is the same for all robots, and the robots are aware of this fact. For this case, we show that the robots can agree on a common x-y coordinate system and elect a leader. We can show that once a unique leader is elected, the robots can be moved to form any geometric figure.

Finally, we consider the case in which the robots are synchronous. It can be shown that in this case, the robots can easily communicate with each other by means of the distances of their moves, once they agree on the unit distance. However, it does not imply that the robots can form any geometric figures, since they may not be able to break certain symmetry in their initial distribution. In fact, whether or not they can form a particular geometric figure depends both on their initial positions and on their local x-y coordinate systems. We therefore consider the problem of determining the class of geometric figures that the robots can form starting from the given initial positions and their local x-y coordinate systems, using the fact that a synchronous robot system can be viewed as an anonymous complete network, which have been investigated, for example, in [11].

We present necessary definitions and basic assumptions in Section 2. Some of the heuristic algorithms proposed previously are reviewed in Section 3. Convergence and formation problems for a point are discussed in Section 4. Agreement on the origin, unit distance, and direction are discussed in Sections 5, 6 and 7, respectively. In Section 8 we consider the case when the robots have a sense of direction. Section 9 considers the case when the robots are synchronous. Concluding remarks are found in Section 10.

2 Definitions and Basic Assumptions

We briefly formalize the problem described in Section 1. Let $r_1, r_2, \ldots, r_n$ be the robots in a two dimensional space. (The subscript "i" of r_i is used for convenience of explanation. The robots do not know their identifiers.) We denote by Z_i, $1 \leq i \leq n$, the local x-y coordinate system of r_i. We assume that it is possible that $Z_i \neq Z_j$ for some i and j. (If $Z_i \neq Z_j$, then Z_i and Z_j do not agree on one or more of the following: the position of the origin, orientation, and unit distance.) As we will see

below, all the positions that r_i observes and computes are given in terms of Z_i.

We assume discrete time $0, 1, 2, \ldots$, and let $p_i(t)$ be the position of r_i at time instant t, where $p_i(0)$ is the initial position of r_i. Define $P(t) = \{p_i(t) | 1 \leq i \leq n\}$ to be the multiset of the positions of the robots at time t. ($P(t)$ is a multiset, since we assume that two robots can occupy the same position simultaneously.) For any point p, we denote by $[p]_j$ the position of p given in terms of Z_j, and define $[P(t)]_j = \{[p_i(t)]_j | 1 \leq i \leq n\}$. Thus $[P(t)]_j$ shows how r_j views the distribution $P(t)$ in terms of its own Z_j. Note that if $Z_j \neq Z_k$, then it is possible that $[P(t)]_j \neq [P(t)]_k$, i.e., r_j and r_k may observe distribution $P(t)$ differently. On the other hand, $[P(t)]_j = [P(t)]_k$ may hold even if $p_j(t) \neq p_k(t)$. In this case, r_j and r_k are located at different positions, but $P(t)$ looks identical to them.

The algorithm that a robot uses is a function ψ such that, for any given sequence $(Q_1, p_1), (Q_2, p_2), \ldots, (Q_m, p_m)$ of pairs of a multiset Q_ℓ of points and a point $p_\ell \in Q_\ell$, $\psi((Q_1, p_1), (Q_2, p_2), \ldots, (Q_m, p_m))$ is a point. Using ψ, we can describe the positions of the robots as follows. At each time instant t, each r_i is either *active* or *inactive*. If r_i is inactive at t, then $p_i(t+1) = p_i(t)$, i.e., r_i does not move. If r_i is active at t, then let $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m = t$ be the time instants when r_i was active, and for each $1 \leq \ell \leq m$, let $Q_\ell = [P(t_\ell)]_i$ and $p_\ell = [p_i(t_\ell)]_i$ be the distribution that r_i observed and the position of r_i at t_ℓ, respectively. (Note that Q_ℓ and p_ℓ are given in terms of Z_i.) Then $p_i(t+1) = p$, where p is the point such that $[p]_i = \psi((Q_1, p_1), (Q_2, p_2), \ldots, (Q_m, p_m))$. That is, r_i moves to point $\psi((Q_1, p_1), (Q_2, p_2), \ldots, (Q_m, p_m))$ of Z_i.

The formalism given above captures the intuition that r_i observes the distribution of the robots only when it is active, and that r_i's next position can depend only on ψ and the distributions that r_i has observed so far. The "p_ℓ" in pair (Q_ℓ, p_ℓ) shows that r_i is always aware of its current position in Z_i. Algorithm ψ is said to be *oblivious* if $\psi((Q_1, p_1), (Q_2, p_2), \ldots, (Q_m, p_m)) = \psi((Q_m, p_m))$ for any $(Q_1, p_1), (Q_2, p_2), \ldots, (Q_m, p_m)$. In this case, the move of a robot depends only on the current configuration of the robots.

Note that the robots are anonymous in the following sense: (1) function ψ is common to all the robots, (2) the identifier "i" of robot r_i is not an argument of ψ, and (3) $[P(t)]_i$ contains only the positions of the robots (but not their identities).

The robots are said to be *synchronous* if every robot is active at every time instant; otherwise, they are *asynchronous*. If the robots are asynchronous, then we assume that every robot becomes active at infinitely many unpredictable time instants. In the following, unless otherwise stated we assume that the robots are not necessarily synchronous.

If the robots are asynchronous, then the robots may not be able to obtain a consistent snapshot of their distribution simultaneously. This, as we will see, is a major technical difficulty in designing correct algorithms. For example, if the robots are synchronous, then all the robots observe their initial distribution simultaneously. So they can adopt the centroid (i.e., the arithmetic mean) of their positions as the common origin, and the minimum nonzero distance between any two robots as the common unit distance, assuming that not all robots are located at the same position

initially. So the robots can move to a point on the circumference of the unit circle centered at the origin, and form an approximation of a circle.

Let π be a predicate over the set of multisets of points that is invariant under any motion (i.e., rotation and parallel transformation) and uniform scaling. For example, π might be true iff the given points are on the circumference of a circle or on a line segment. For such π, we consider two types of problems, the *convergence problem* and the *formation problem.* In the convergence problem, the goal is to design an algorithm ψ such that, as t goes to infinity, $P(t)$ converges to a distribution that satisfies π, regardless of the number n of robots, the initial distribution $P(0)$, and (if the robots are not necessarily synchronous) which robots are active at each time instant. The goal of the formation problem is similar, except that the robots must reach some points satisfying π in finite steps and "halt". That is, there must exist some time instant t' such that $P(t')$ satisfies π and $p_i(t') = p_i(t'+1) = \cdots$ for all $1 \leq i \leq n$. Since the robots have no knowledge of the underlying coordinate system that we use for describing π, all we can expect is to have the robots converge to or form a figure similar to the given goal figure. The restriction on π stated above was introduced for this reason. All predicates we discuss in the following satisfy this condition.

In addition to convergence and formation problems for a predicate, we discuss *agreement problems* for a given concept C, where C might be a location, length or direction. Unfortunately, it is not very convenient to do so within the framework introduced above, since the only property of the robots that are directly observable to us is their movement. So we extend the framework minimally as follows. We imagine that each robot r_i has a *local variable* α_i, whose value is *undefined* at time 0. The problem is to design a deterministic algorithm ψ that computes the new positions of a robot *and* the values to be assigned to its local variable, such that, in any scenario that arises under ψ, there exists some time instant t_0 such that

1. for each $1 \leq i \leq n$, the value of α_i is defined at t_0 and remains unchanged after t_0, and

2. the values of $\alpha_1, \alpha_2, \ldots, \alpha_n$ at t_0 "agree" on concept C.

For example, if C is location, then the second condition requires that each α_i is a position p_i of Z_i and $p_1, p_2, \ldots, p_n$ all refer to the same point p, i.e., $[p]_i = p_i$, $1 \leq i \leq n$, for some point p. (We can define a similar requirement for agreement for other concepts or combinations of concepts.) This definition of agreement is weaker than that of "common knowledge" that requires "Everyone knows that everyone knows that ... that everyone knows it" for arbitrary depth [3]. It is well-known that if the robots are not synchronous, then they cannot obtain common knowledge that does not exist initially. Leader election is also viewed as an agreement problem, but for this case we require that after some time t_0, $\alpha_i = 1$ for exactly one robot r_i (the leader) and $\alpha_j = 0$ for all other robots r_j, $j \neq i$.

In this paper, we do not consider dynamic changes in the number of robots while an algorithm is executed. However, using this framework, it is possible to discuss the

situation in which some robots are added and/or removed from the system dynamically. By definition, an oblivious algorithm correctly solves the given problem even if the number of robots changes a finite number of times. For non-oblivious algorithms, we need additional assumptions. Assume that a robot becomes visible (or invisible) when it is added (or removed) from the system, and that if the number of robots changes, then it never changes again until all the robots have noticed the change. Now, modify the given algorithm so that a robot executing it "resets its memory and restarts" (i.e., it ignores the pairs (Q_ℓ, p_ℓ) for the observations made previously) if it notices a change in the number of robots. Under these assumptions, the modified algorithm correctly works even if the number of robots changes a finite number of times. An algorithm having this property can be viewed as a self-stabilizing algorithm, since it solves the given problem in the presence of transient failures.[2] This is an advantage of the distributed control method. In the centralized method, the entire system can crash if the robot controlling all other robots becomes faulty (and is removed).

3 Heuristic Algorithms

Before starting a formal study of algorithms, we first review some known heuristic algorithms for forming a circle and a line segment. These algorithms commonly contain a phase to intersperse the robots more uniformly on a circle or a line segment by moving each robot away from its nearest neighbor, but for simplicity, we omit it in the descriptions given below.

In the following, for robot r_i, furthest(r_i) (resp. nearest(r_i)) is any of the robots furthest (resp. nearest) from r_i in the current configuration. (Ties can be broken using any deterministic method.) ν is the maximum distance a robot can move at a time. It is assumed that the robots have a common sense of unit distance.

In the following, when presenting an algorithm, for convenience of discussion we give an informal description of the behavior of a robot or robots executing it, instead of giving a formal definition of function ψ. Converting the informal description into a formal definition is straightforward and tedious.

Algorithm $\psi_{circle1}$ given next was proposed by Sugihara and Suzuki [7] for converging the robots to a circle with radius a, for given a. A detailed investigation of the algorithm is found in Tanaka et al. [9].

Algorithm $\psi_{circle1}$
Each time r_i becomes active, it calculates the distance d to furthest(r_i). If $d-2a \geq 0$, then it moves distance $\min\{d-2a, \nu\}$ towards furthest(r_i). If $d-2a < 0$, then it moves distance $\min\{2a-d, \nu\}$ away from furthest(r_i). □

[2] A change in the number of robots can be viewed as a transient failure that damages the memory of a robot, except the information on the current distribution of the robots. Since any robot can correctly observe the current distribution, here the term self-stabilizing is used in a weaker sense than the standard one [6].

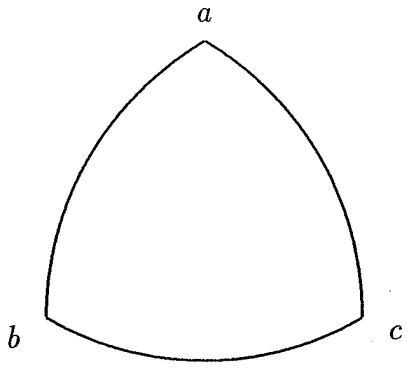

Figure 1: Reuleaux's triangle.

As is pointed out in [7], the robots using $\psi_{circle1}$ sometimes converge to a Reuleaux's triangle (Figure 1), but this small flaw can be remedied considerably by tuning the value of ν [9]. Further, Tanaka [8] recently proposed a new algorithm $\psi_{circle2}$ given below, and showed, using simulation, that it works better than $\psi_{circle1}$, avoiding convergence to a Reuleaux's triangle when n is large.

Algorithm $\psi_{circle2}$
Each time r_i becomes active, it calculates the distance d to the middle point M of nearest(r_i) and furthest(r_i). If $d-a \geq 0$, then it moves distance $\min\{d-a, \nu\}$ towards M. If $d - a < 0$, then it moves distance $\min\{a - d, \nu\}$ away from M. □

As for the problem of forming a line segment, Hirota recently proposed the following algorithm ψ_{line} [4].

Algorithm ψ_{line}
Each time r_i becomes active, it calculates the distance d to the point p that is the foot of the perpendicular drop from r_i to the line ℓ passing through nearest(r_i) and furthest(r_i). Then it moves $\min\{d, \nu\}$ towards p. □

These algorithms are oblivious and surprisingly simple. This fact seems to demonstrate the usefulness and potential of the distributed approach for controlling the robots. However, the main emphasis of the works mentioned above is on the development of heuristic algorithms and presentation of simulation results, and formal discussions on the correctness and performance of the algorithms are not given. In what follows, we study such distributed algorithms formally. Since the behavior of the robots can be more complex than we might expect, we start the investigation with a very simple problem of converging the robots to a point.

4 Convergence and Formation Problems for a Point

We start with two mutually related problems of converging the robots to a point and moving the robots to a point. These problems are perhaps some of the simplest

problems one could consider. Nevertheless, the discussions presented in this section can serve as an introduction to the technical results given in the rest of the paper.

4.1 Converging to a Point

Formally, the problem of converging the robots to a point is stated as the convergence problem for predicate π_{point}, where $\pi_{point}(p_1, \ldots, p_n) =$ true iff $p_i = p_j$ for any $1 \leq i, j \leq n$. We call this problem POINT. We present two oblivious algorithm for POINT and some incorrect variations of one of them.

The following algorithm ψ_{point1} moves each robot r_i toward furthest(r_i), so that the distance between them will be reduced either by a constant factor, or by a constant.

Algorithm ψ_{point1}
Each time r_i becomes active, it calculates the distance d to furthest(r_i) and moves distance $\min\{d/b_i, \nu_i\}$ towards furthest(r_i). Here, $b_i > 1$ is a constant and $\nu_i > 0$ is the maximum distance r_i can move at a time. (b_i and ν_i need not be the same for all robots.) □

Remark 1 Algorithm ψ_{point1} is oblivious. Also, in ψ_{point1}, d and ν_i are given in terms of Z_i.

Theorem 1 *Algorithm ψ_{point1} correctly solves problem POINT.*

Proof Recall that $P(t)$ denotes the distribution of the robots at time instant t. It suffices to show that $CH(P(t))$ converges to a single point as t goes to infinity, where CH denotes the convex hull. Suppose that $CH(P(t))$ does not converge to a single point. Then since $CH(P(t)) \supseteq CH(P(t+1))$ for any time instant t, $CH(P(t))$ must converge to a convex polygon C. Let D be the radius of a largest circle that fits in C. (If C is a line segment, then let D be one half of the length of C.) Since $CH(P(t)) \supseteq C$ for any t, the distance between a robot and its furthest neighbor is always at least D. So if we choose a constant $\epsilon > 0$ such that $\epsilon < \min_{1 \leq i \leq n}\{D/b_i, \nu_i, D/(1 - 1/b_i)\}$, then when any robot r_i located at point p moves toward its furthest neighbor located at point q, the new position of r_i is at distance greater than ϵ from both p and q. Then, since C is a convex polygon, there exists a sufficiently small constant $\delta > 0$ (that depends on ϵ and C) such that, if the new position of r_i is in the δ-neighborhood δ_v of a corner v of C, then at least one of p and q (defined above) is not in the δ-neighborhood δ_C of C. Now, since δ can be made arbitrarily small, we may choose δ so that $\delta < \epsilon/2$. Then, since $CH(P(t))$ converges to C, there must exist some t_0 such that for any $t \geq t_0$, (1) every robot is in δ_C at t and (2) there is at least one robot in δ_v of every corner v of C at t. However, we can show that the number of robots in δ_v monotonically decreases after t_0, a contradiction to condition (2). To see this, note that for every corner v of C, every robot r_i at point p in δ_v eventually moves, and when it moves, it leaves the ϵ-neighborhood ϵ_p of p. So it leaves δ_v since $\delta < \epsilon/2$ and $p \in \delta_v$. Suppose r_i enters δ_v after t_0. Then as we mentioned above, either r_i or its furthest neighbor must have been outside δ_C before the move, a contradiction to condition (1). □

Suppose we modify ψ_{point1}, so that $b_i = 1$ for all the robots. This means that robot r_i moves to the position of its furthest neighbor, if that neighbor is located within the maximum distance that r_i can move in one step. The modified algorithm does not solve POINT. For example, if (1) there are only two robots and either can move to the position of the other in one move, and (2) they happen to be synchronous, then they will continue to swap their positions forever. Also, ψ_{point1} fails to solve POINT if d is defined to be the distance to a *nearest* neighbor of r_i. To see this, consider the case of four robots r_1, r_2, r_3 and r_4 in which, initially, r_1 and r_2 are close to each other, r_3 and r_4 are close to each other, but the r_1-r_2 pair and the r_3-r_4 pair are far apart. Suppose that only one robot becomes active at a time. Then, r_1 and r_2 converge to a point, and r_3 and r_4 converge to a point, but since this process never terminates in finite steps, the four robots never converge to a point.

Another correct algorithm for POINT is the following. Note that this algorithm is also oblivious.

Algorithm ψ_{point2}
Each time r_i becomes active, it calculates the distance d to the centroid g of the positions of the robots, and moves distance $\min\{d/b_i, \nu_i\}$ towards g. Here, $b_i > 1$ is a constant and $\nu_i > 0$ is the maximum distance r_i can move at a time. (b_i and ν_i need not be the same for all robots.) □

We can prove the correctness of ψ_{point2} using an argument similar to that in the proof of Theorem 1. The proof is omitted to save space.

4.2 Moving to a Point

Next, we discuss the formation problem for predicate π_{point} introduced in Subsection 4.1. We call this problem MEET. Since MEET is a formation problem, all the robots must move to a single point *in finite steps.* The next theorem states that problem MEET cannot be solved by any oblivious algorithms, even if the number n of robots is two. Recall that functions ψ_{point1} and ψ_{point2} of Subsection 4.1 are *oblivious* algorithms for solving the corresponding convergence problem POINT.

Theorem 2 *There is no oblivious algorithm for solving MEET, even for the case* $n = 2$.

Proof Suppose that there is an oblivious algorithm ψ that solves MEET for two robots r_i and r_j. Note that since ψ is oblivious, the moves of the robots depend only on Z_i, Z_j and their current positions. We first show that there exist distinct positions p and q of r_i and r_j, respectively, such that either (1) ψ moves r_i from p to q, and r_j from q to q, or (2) ψ moves r_i from p to p, and r_j from q to p. (That is, ψ moves exactly one robot to the position of the other, if both robots become active simultaneously.) To see this, assume that such positions do not exist. Consider a scenario $\mathcal{S}$ in which r_i and r_j, located at distinct positions p and q, respectively, at

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Figure 2: (a) r_i moves but r_j does not. (b) After modification of Z_i.

time $t-1$, occupy the same position r at time t. Now we show that we can modify this scenario and obtain another scenario in which the robots never occupy the same position simultaneously. There are two cases.

Case 1: Both r_i and r_j are active at time $t-1$ in $\mathcal{S}$.
By assumption, $r \neq p$ and $r \neq q$. So if exactly one robot, say r_i, happens to be active at $t-1$, then at time t, r_i is located at r and r_j at q, where $r \neq q$.

Case 2: Exactly one robot is active at $t-1$ in $\mathcal{S}$.
Suppose that r_i is active at $t-1$ but r_j is not. Then $r = q$. So if both robots happen to be active at $t-1$, then at time t, r_i is located at q and r_j at some point s, where by assumption, $s \neq q$.

Using this argument repeatedly, we can construct an infinite sequence of moves in which the robots never occupy the same position simultaneously. (We can do so in such a way that each robot becomes active infinitely many times, since either of the robots can be chosen to be inactive in Case 1.) So ψ does not solve MEET. This is a contradiction.

So consider an initial distribution $P(0) = \{p, q\}$, where $p \neq q$, in which r_i and r_j are at p and q, respectively, and ψ moves r_i from p to q, and r_j from q to q. See Figure 2(a). (The case in which ψ moves r_j to the positions of r_i is similar.) Now, by modifying Z_i through translation and rotation, we can construct another configuration in which r_i observes distribution $P(0)$ the same way as r_j does, i.e., $[P(0)]_i = [P(0)]_j$ and $[p]_i = [q]_j$. See Figure 2(b). Then ψ moves both r_i and r_j in the same manner in the new configuration, and of course, ψ moves r_j in the same manner in both configurations (namely, from q to q). Therefore in the new configuration, ψ moves r_i from p to p, and r_j from q to q. Then, since ψ is oblivious, both robots remain in their respective initial positions forever. So ψ does not solve MEET. This is a contradiction. □

Theorem 2 shows a limitation of oblivious algorithms. Although oblivious algorithms are easy to understand and analyze, they may not be powerful enough to achieve certain goals. For example, intuitively, the robots' ability to communicate with each other could be severely limited, since memorizing some common concepts (e.g., the size of the unit distance, the direction of north) seems to be inevitable for effective communication.

On the other hand, MEET can be solved for two robots by a *non-oblivious* algorithm. One such algorithm is $\psi_{meet(2)}$ given next. Algorithm $\psi_{meet(2)}$, as well as all other algorithms in the rest of the paper, has been written under the assumption

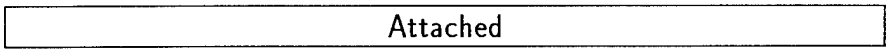

Figure 3: Illustration for $\psi_{meet(2)}$.

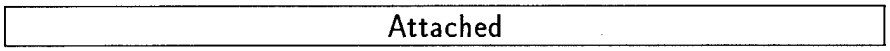

Figure 4: The case $t_i = t_j$.

that the robots occupy *distinct* positions in the initial distribution. In Appendix A, we give an algorithm for transforming any given distribution of any number of robots (in which some robots may occupy the same position) into one in which (1) no two robots occupy the same position, and (2) the robots are not located on a single line segment if $n > 2$, under the assumption that clones do not exist.

Algorithm $\psi_{meet(2)}$
When r_i becomes active for the first time, it translates and rotates its coordinate system[3] Z_i so that

1. r_i is at $(0,0)$ of Z_i, and

2. the other robot r_j is on the positive y-axis of Z_i, say at $(0,a)$ for some $a > 0$.

Then it moves in the positive x direction of Z_i, over any nonzero distance. It then continues to move in the same direction each time it becomes active, until it observes that the position of r_j has changed twice. (See Figure 3.)

Now, r_i knows line ℓ that contains the first two distinct positions of r_j that r_i has observed. (Note that by symmetry, ℓ is the x-axis of r_j's coordinate system Z_j.) Then using Lemma 1 given below, r_i finds the initial position of r_j, and moves to the midpoint of the initial positions of r_i and r_j. □

Lemma 1 shows that robots r_i and r_j executing $\psi_{meet(2)}$ eventually find out which of them became active first for the first time, and what their initial distribution was.

Lemma 1 *Let t_i and t_j be the time instants at which r_i and r_j, respectively, become active for the first time in $\psi_{meet(2)}$. Then the following hold.*

1. *The trajectory of r_i and the trajectory of r_j are parallel iff $t_i = t_j$. In this case, each robot sees the other robot at its initial position at $t_i(= t_j)$ (Figure 4).*

[3]Formally, r_i cannot modify Z_i in our framework. For convenience of explanation, however, we sometimes imagine that r_i transforms Z_i into a new coordinate system. The effect of such a transformation can easily be simulated within the framework in which Z_i remains unchanged.

Attached

Figure 5: The case $t_i < t_j$.

2. *The trajectory of r_j intersects the negative x-axis of Z_i iff $t_i < t_j$. In this case, r_i sees r_j at its initial position, and r_i's initial position is the foot of the perpendicular drop from r_j's initial position to the line containing the trajectory of r_i (Figure 5).*

3. *The trajectory of r_i intersects the negative x-axis of Z_j iff $t_j < t_i$. In this case, r_j sees r_i at its initial position, and r_j's initial position is the foot of the vertical drop from r_i's initial position to the line containing the trajectory of r_j.*

Proof The lemma follows immediately from the description of $\psi_{meet(2)}$. □

Theorem 3 *Algorithm $\psi_{meet(2)}$ solves problem MEET for the case $n = 2$.*

Proof The key observation is the following: When r_i observes that the position of r_j has changed twice, r_j must have already observed that r_i's position has changed at least once, and thus r_j knows where the x-axis of Z_i is. Similarly, r_j will know that r_i knows where the x-axis of Z_j is. Then the correctness of $\psi_{meet(2)}$ follows from Lemma 1. □

Before presenting a non-oblivious algorithm for solving MEET for three or more robots, we discuss a technique that we use often for designing algorithms. One technical difficulty is that in general, a robot cannot determine, given the positions of the robots observed at two time instants t_1 and t_2, which robot has moved to which position between t_1 and t_2. One way to overcome this problem is to impose a bound on the maximum distance that any robot can move while other robots remain inactive, so that the robot at position p at time t_1 must be at the position closest to p at time t_2. Specifically, we do the following. When robot r_i becomes active for the first time, it memorizes the distance $a_i > 0$ to its nearest neighbor. Then r_i moves at most distance $a_i\epsilon/2^k$ in the k-th move, where $0 < \epsilon \leq 1/2$ is a constant chosen by the algorithm. This restriction assures that r_i remains in the interior of the $a_i\epsilon$-neighborhood of its initial position. Since the interiors of such neighborhoods of two robots located at different positions do not intersect, any robot can correctly know the position of r_i, even after it remains inactive for a long time. This technique and its variations are used in ψ_{meet} given next, $\psi_{scatter}$ of Appendix A, and some other algorithms.

The following is a non-oblivious algorithm ψ_{meet} for solving MEET for three or more robots. It is assumed that in the initial distribution, (1) no two robots occupy the same position, and (2) the robots are not located on a single line segment. Appendix A explains how a given distribution can be transformed into one satisfying these conditions.

Attached

Figure 6: Illustration for ψ_{meet}.

Algorithm ψ_{meet}
When r_i becomes active for the first time, it determines whether or not it is located at a corner of the convex hull C of the distribution of the robots at that time instant. There are two cases.

Case 1: r_i is not at a corner of C.
r_i memorizes the position p of its nearest neighbor and the distance a_i to p. Then it moves towards p, and continues to move toward p each time it subsequently becomes active, staying in the interior of the $(a_i/2)$-neighborhood of its initial position. (See the explanation given above.)

Case 2: r_i is at a corner of C.
Let a, b, c and d be consecutive corners of C in clockwise order, where r_i is at b. See Figure 6. Then r_i memorizes the direction that is away from a along the line containing $\overline{ab}$, and moves in that direction each time it becomes active, staying in the interior of the $a_i\epsilon$-neighborhood of b for some $\epsilon \leq 1/2$. Here, ϵ is chosen so that the $a_i\epsilon$-neighborhood does not intersect the line containing $\overline{cd}$. (This assures that the robot r_j at corner c remains to be at a corner of the convex hulls of the new positions of the robots even after r_i and r_j move.)

Robot r_i continues to move as described above, until it observes that the position of each robot has changed at least twice. Then, r_i knows line ℓ_j that contains the first two distinct positions of r_j that r_i has observed, for each robot r_j at a corner of the initial convex hull. Since the convex hull of the initial positions of the robots is a k-sided polygon for some $k \geq 3$, the lines $\{\ell_j\}$ determine a unique, smallest convex polygon Q that contains the initial convex hull. Then r_i moves to the centroid of the corners of Q. □

Theorem 4 *Algorithm ψ_{meet} solves MEET for $n > 2$.*

Outline of Proof Using an argument similar to the one in the proof of Theorem 3, we can show that every robot executing ψ_{meet} eventually knows the lines $\{\ell_j\}$ correctly. (The trajectories of the robots that are not at a corner of the initial convex hull are not used to define Q. We move such robots, however, so that other robots r_i will know that they have become active sufficiently many times and observed r_i's movement.) So all the robots eventually know the corners of Q. Thus ψ_{meet} solves MEET for $n > 2$. □

By Theorems 3 and 4, MEET is solvable for any $n \geq 2$.

5 Agreement on the Origin

In this section we discuss the problem of agreement on the origin of a common x-y coordinate system. We call this problem ORIGIN. ORIGIN is an agreement problem for concept C as defined in Section 2, where C is a location.

Theorem 5 *ORIGIN is solvable for any $n \geq 2$.*

Proof As we have seen, the robots that are initially at distinct positions and executing $\psi_{meet(2)}$ (for the case $n = 2$) or ψ_{meet} (for the case $n > 2$) eventually know the point at which they meet (the midpoint of the initial positions for the case $n = 2$, and the centroid of the corners of Q for the case $n > 2$). So they can agree on that point. □

6 Agreement on the Unit Distance

We call the problem of agreeing on the unit distance UNITDIST. This is an agreement problem for concept C, where C is a length. So the goal is to let each robot r_i decide on a length L_i in terms of its own Z_i, so that the lengths $L_1, L_2, \ldots, L_n$ chosen by the robots all refer to the same physical length.

Theorem 6 *UNITDIST is solvable for any $n \geq 2$.*

Proof For the case $n = 2$, using $\psi_{meet(2)}$, each robot eventually finds the initial position of the other (Lemma 1). Then they can use the distance between their initial positions as the unit distance. For the case $n > 2$, the robots can choose, as the unit distance, the length of a shortest side of the k-sided convex polygon Q that they obtained using algorithm ψ_{meet} in Subsection 4.2. □

The proofs of Theorems 5 and 6 show that the robots can agree on both a point and a length simultaneously. Thus they can agree on a circle whose center is at the agreed point and whose radius is the agreed length. So we have:

Corollary 1 *The problem of agreeing on a circle and the problem of forming an approximation of a circle (in the sense that all the robots are located on the circumference of a common circle) are solvable for any $n \geq 2$.*

7 Agreement on Direction

Now we show that the third problem of agreeing on direction is unsolvable in general. Let us call this problem DIRECTION. Here, the concept C on which the robots agree is a direction.

Figure 7: A symmetric configuration.

Theorem 7 *There is no algorithm for solving DIRECTION, even if $n = 2$ and the robots are synchronous.*

Proof Consider two synchronous robots r_i and r_j such that initially,

1. r_i is located at position $(0,0)$ of Z_i and at position $(0,1)$ of Z_j, and
2. r_j is located at position $(0,0)$ of Z_j and at position $(0,1)$ of Z_i.

See Figure 7. Then the robots always move in the same (symmetric) manner, and thus when r_i chooses a direction, r_j chooses the opposite direction. □

Note that as we stated in Section 2, the other two agreement problems, ORIGIN and UNITDIST, are trivially solvable if the robots are synchronous.

Using an argument, similar to that in the proof of Theorem 7, involving three robots that initially form an equilateral triangle, we can prove the following theorem:

Theorem 8 *The problem of forming a line segment is unsolvable.* □

8 Robots with a Sense of Direction

We say that the robots have a sense of direction if they agree on the direction of the positive x-axis, which we call "east." (We use "west," "north," and "south" in the understood manner.) The main result of this section is the following.

Theorem 9 *If the robots have a sense of direction, then they can agree on a common coordinate system, and elect a leader.*

Proof The theorem follows from the discussion given below. □

Again, it is assumed that initially, (1) no two robots are located at the same position, and (2) the robots are not located on a single line segment if $n > 2$.

The argument for the case $n = 2$ is similar to those in the proofs of Theorems 5 and 6. Using $\psi_{meet(2)}$, robots r_i and r_j can adopt the midpoint of their initial positions as the origin, and the distance between their initial positions as the unit distance. Then, since the robots have a sense of direction, they have agreed on a coordinate system. Finally, the robot whose initial position in the agreed coordinate system is larger in lexicographic ordering can be selected as the leader.

For the case $n > 2$, we present a new algorithm, $\psi_{coordinate}$, that solves *both* the coordinate agreement problem and the leader election problem, when the robots have a sense of direction. (Of course, the possibility of agreement on a coordinate system alone follows immediately from the discussions in Sections 5 and 6.)

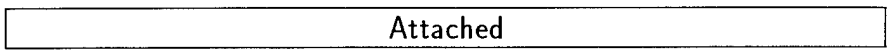

Figure 8: Algorithm $\psi_{coordinate}$.

Algorithm $\psi_{coordinate}$
The idea is to choose (1) the vertical (i.e., north-south) line though the eastern-most robots as the common y-axis, (2) the distance between the two vertical lines, one through the eastern-most robots and the other through the western-most robots, as the unit distance, and (3) the horizontal (i.e., east-west) line through the northern-most robots (excluding two special robots, as explained below) as the common x-axis. To achieve this, the robots do the following.

1. The northern-most robot among the eastern-most robots continues to move north within a "small" neighborhood of its initial position (as explained in Section 4.2). All other eastern-most robots continue to move west within a "small" neighborhood of its initial position, in such a way that they will never become western-most.

2. The northern-most robot among the western-most robots continues to move north within a "small" neighborhood of its initial position. All other western-most robots continue to move east within a "small" neighborhood of its initial position, in such a way that they will never become eastern-most.

3. The robots that are neither eastern-most nor western-most continue to move west within a "small" neighborhood of its initial position, in such a way that they never become western-most.

See Figure 8. Then eventually, every robot knows the trajectories of all other robots, and chooses the line containing eastern vertical trajectory as the common y-axis, and the line containing the northern-most horizontal trajectory as the common x-axis. The unit distance is chosen to be the distance between the two vertical trajectories. The leader is the robot moving along the eastern vertical trajectory. □

The correctness of $\psi_{coordinate}$ should be immediate from its description, once we note that, since $n > 2$ and the initial distribution of the robots satisfy the assumptions given above, the two vertical trajectories do not overlap (so the unit distance is well-defined) and there is at least one horizontal trajectory (so the common x-axis is well-defined). So Theorem 9 follows.

Since the leader can compute the final positions of all the robots that satisfy any given predicate and "guide" them to their respective final positions, we obtain the following theorem.

Theorem 10 *If the robots have a sense of direction, then for any predicate π that is invariant under any motion and uniform scaling, the convergence and formation problems for π are solvable.*

Proof Let π be any given predicate that is invariant under any motion and uniform scaling. We only need to show how the formation problem for π can be solved. First, using $\psi_{coordinate}$, the robots elect the leader, say robot r_i. Even after r_i is elected, the robots except r_i continue to move within their respective "small" neighborhoods of their initial positions, as specified in $\psi_{coordinate}$. Then r_i computes a final position p_j for every robot r_j, such that p_j is to the east of r_i's own vertical trajectory and the multiset $\{p_j\}$ satisfies π. (Recall that π is invariant under any motion. Also, since r_i is one of the eastern-most robots, the final positions p_j are all to the east of the current positions of the robots.) Now, r_i "guides" all other robots to their respective new positions one by one. Specifically, r_i selects one robot r_j at a time and moves to the current position of r_j. It is possible that r_j moves to a new position simultaneously, but r_i is now on the trajectory of r_j and within the "small" neighborhood that r_j uses. Since no other robot can enter this region, r_j knows that r_i has selected r_j. When r_i observes that r_j moves again, r_i knows that r_j knows that r_j has been selected. So r_i moves to p_j and waits until r_j moves to that point. (Meanwhile, r_j moves to the position where r_i has moved to. Here, if it takes two or more steps for r_i to reach p_j, then r_i moves monotonically eastbound to p_j, so that r_j knows that r_j has reached p_j when r_i moves westbound after r_j catches up with r_i.) Robot r_i does this repeatedly for all other robots, and then moves to its own final position p_i. □

In particular, from Theorem 10 we have:

Corollary 2 *If the robots have a sense of direction, then the problem of forming a line segment is solvable.*

9 Synchronous Robots

In this section, we characterize the the class of geometric figures that the robots can form, under the assumption that they are synchronous. Again, we assume that initially, all robots occupy distinct positions. In addition, for simplicity of explanation, we assume that (the robots know that) each robot r_i is located at the origin of its coordinate system Z_i at time 0. Essentially the same results hold even without this assumption.

What geometric figures can be formed depends not only on the given initial positions of the robots, but also on their local x-y coordinate systems. For example, suppose that initially, four robots r_1, r_2, r_3 and r_4 form a square in counterclockwise order, where r_2 is at position $(1,0)$ of Z_1, r_3 is at position $(1,0)$ of Z_2, and so on. See Figure 9. Then the robots have the same "view", and since the clocks are synchronized and the algorithm they use is deterministic, they can never break symmetry (and intuitively, they continue to form a square all the time). On the other hand, if the direction of the positive x-axis happens to be the same for all four robots (Figure 10), then the robots can easily discover this fact and elect the robot that is northern-most among the eastern-most robots as the leader (and form any geometric

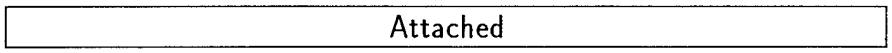

Figure 9: A symmetric configuration of four robots.

Figure 10: A configuration that is not symmetric.

figure). Intuitively, in this case every robot has a unique "view," and thus the robots can elect a leader using a suitable total ordering of the "views." In the following, we formalize this observation.

Following [11], the *view* of robot r_i at time t, denoted $V_i(t)$, is defined recursively as a rooted infinite tree as follows.

1. The root of $V_i(t)$ has $n-1$ subtrees, one for each robot r_j, $j \neq i$.
2. The edge from the root of $V_i(t)$ to the subtree corresponding to r_j is labelled $((a,b),(c,d))$, where (a,b) is the position of r_j in terms of Z_i, and (c,d) is the position of r_i in terms of Z_j.
3. The subtree corresponding to r_j is the view $V_j(t)$ of r_j at time t.

Note that each vertex of $V_i(t)$ corresponds to a robot, but it is not labelled as such. Two views $V_i(t)$ and $V_j(t')$ are said to be *equivalent*, written $V_i(t) \equiv V_j(t')$, if they are isomorphic to each other including the labels.

$V_i(0)$ is thus the view of r_i when it becomes active for the first time. Note that since the robots occupy distinct positions at time 0, the edges incident on the root of $V_i(0)$ have distinct labels. Since at time 0 the robots have no knowledge of other robots' local coordinate systems, at time 0 robot r_i does not know its view $V_i(0)$.

It is possible for the robots to obtain sufficient information to construct their views. The following algorithm achieves this.

Algorithm $\psi_{getview}$
At time 0, the robots adopt the minimum distance between any two robots as the common unit distance. Then each robot r_i moves in the direction of its positive x-axis over distance $f(d_i)/2$ measured in terms of the common unit distance, where $d_i > 0$ is the common unit distance measured in terms of Z_i and for $x > 0$, $f(x) = 1 - 1/2^x$ is a monotonically increasing function with range $[0,1)$. When the robots become active at time 1, they all return to their respective initial positions.[4] □

[4]If the maximum distance that r_i can move in one step is smaller than $f(d_i)/2$, then we can let r_i continue to move in the same direction until the total distance of the moves is exactly $f(d_i)/2$. Since the robots are synchronous, they can wait until all robots have moved over the desired distances and stop, and then return to their initial positions, possibly using the same technique.

At time 1, r_i knows, for each robot r_j, the direction of the positive x-axis and the unit distance of Z_j. Thus, since by assumption the origin of Z_j is the same as the initial position of r_j, r_i knows Z_j and the the positions of all the robots in terms of Z_j. Using this information, r_i can construct its view $V_i(0)$. Note that since each robot returns to its initial position when it moves at time 1, we have $V_i(0) \equiv V_i(2)$.

Let m be the size of a largest subset of robots having an equivalent view at time 0. If $m = 1$, then every robot has a unique view, and thus once Algorithm $\psi_{getview}$ is executed, using a suitable total ordering over the views, the robot having the largest view can be chosen as the leader, and thus the robots can form any geometric figure, as explained in the proof of Theorem 10. So in the following, we consider the case $m \geq 2$. Lemmas 2, 3, 4 and 5 given below refer to a fixed initial configuration with $m \geq 2$.

Lemma 2 *The robots can be partitioned into n/m groups of m robots each, such that two robots have an equivalent view iff they belong to the same group.*

Proof The claim is trivial if $m = n$. So assume that $m < n$, and without loss of generality suppose that $V_1(0) \equiv V_2(0) \equiv \cdots \equiv V_m(0)$ but $V_1(0) \not\equiv V_{m+1}(0)$. That is, $r_1, r_2, \ldots, r_m$ have an equivalent view at time 0 but r_{m+1} does not. Let $((a,b),(c,d))$ be the label of the edge from the root of $V_1(0)$ to the vertex corresponding to r_{m+1}. Since $V_1(0) \equiv V_2(0) \equiv \cdots \equiv V_m(0)$, for each ℓ, $1 \leq \ell \leq m$, there exists an edge with label $((a,b),(c,d))$ from the root of $V_\ell(0)$ to a vertex corresponding to some robot r_{i_ℓ}, where $r_{i_1} = r_{m+1}$. Now we show that the robots $r_{i_1}, r_{i_2}, \ldots, r_{i_m}$ are all distinct. To see this, note that by symmetry, there is an edge with label $((c,d),(a,b))$ from the root of $V_{i_\ell}(0)$, leading to a vertex that corresponds to robot r_ℓ. So if $r_{i_1} = r_{i_2}$, for instance, then we have $r_1 = r_2$, a contradiction. Thus $r_{i_1}, r_{i_2}, \ldots, r_{i_m}$ are all distinct. Furthermore, since $V_1(0) \equiv V_2(0) \equiv \cdots \equiv V_m(0)$ and $V_{i_\ell}(0)$ is a subtree of $V_\ell(0)$ connected to the root of $V_\ell(0)$ by an edge with label $((a,b),(c,d))$ for each ℓ, we have $V_{i_1}(0) \equiv V_{i_2}(0) \equiv \cdots \equiv V_{i_m}(0)$. Thus there are at least m robots (including r_{m+1}) having a view equivalent to that of r_{m+1}. But then, there must be exactly m such robots, since there cannot exist more than m such robots by the definition of m. The lemma follows from this observation. □

Lemma 3 *At time 0, the robots in the same group form a regular m-gon, and the regular m-gons formed by all the groups have a common center.*

Proof Suppose that $V_1(0) \equiv V_2(0) \equiv \cdots \equiv V_m(0)$, that is, $r_1, r_2, \ldots, r_m$ have an equivalent view at time 0. Consider the initial positions $p_1(0), p_2(0), \ldots, p_m(0)$ of these robots. Clearly at least one of $p_1(0), p_2(0), \ldots, p_m(0)$ is a corner of the convex hull C of $\{p_1(0), p_2(0), \ldots, p_m(0)\}$. Then, since $V_1(0) \equiv V_2(0) \equiv \cdots \equiv V_m(0)$, each of $p_1(0), p_2(0), \ldots, p_m(0)$ must be a corner of C. Without loss of generality assume that $p_1(0), p_2(0), \ldots, p_m(0)$ occur in counterclockwise order around the convex hull. (See Figure 11.) Since $V_1(0) \equiv V_2(0) \equiv \cdots \equiv V_m(0)$, the internal angles of C at the corners $p_1(0), p_2(0), \ldots, p_m(0)$ must be all identical, and the lengths of the

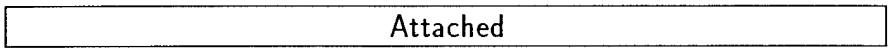

Figure 11: Illustration for the proof of Lemma 3, for the case $m = 4$.

edges of the convex hull must be all identical. (If $\overline{p_1(0)p_2(0)}$ looks shorter than $\overline{p_2(0)p_3(0)}$ to r_2, then $\overline{p_2(0)p_3(0)}$ should look shorter than $\overline{p_3(0)p_4(0)}$ to r_3, and so on, leading to a conclusion that $\overline{p_1(0)p_2(0)}$ is shorter than $\overline{p_1(0)p_2(0)}$, a contradiction.) So $p_1(0), p_2(0), \ldots, p_m(0)$ form a regular m-gon.

Suppose that at time 0, $r_{m+1}, r_{m+2}, \ldots, r_{2m}$ also have a view that are mutually equivalent, and that their respective positions $p_{m+1}(0), p_{m+2}(0), \ldots, p_{2m}(0)$ appear in counterclockwise order around the regular m-gon they form. Then again, since $V_1(0) \equiv V_2(0) \equiv \cdots \equiv V_m(0)$, the position of $p_{m+1}(0)$ relative to p_1 is the same as the position of $p_{m+2}(0)$ relative to p_2, and so on. (See Figure 11.) So the regular m-gon formed by $p_1(0), p_2(0), \ldots, p_m(0)$ and the regular m-gon formed by $p_{m+1}(0), p_{m+2}(0), \ldots, p_{2m}(0)$ have the same center. □

Lemma 4 *For any (deterministic) algorithm ψ, at any time instant t, the robots in the same group form a regular m-gon, and the regular m-gons formed by all the groups have a common center.*

Proof Suppose that $V_1(0) \equiv V_2(0) \equiv \cdots \equiv V_m(0)$, that is, $r_1, r_2, \ldots, r_m$ have an equivalent view at time 0. Now, since the initial distribution of the robots looks identical to $r_1, r_2, \ldots, r_m$, the new positions they compute using ψ and their respective $Z_1, Z_2, \ldots, Z_m$ are all identical. Also, since $V_1(0) \equiv V_2(0) \equiv \cdots \equiv V_m(0)$, the center of the regular m-gon that $r_1, r_2, \ldots, r_m$ form at time 0 has the same x-y coordinates in all of $Z_1, Z_2, \ldots, Z_m$. This means that $r_1, r_2, \ldots, r_m$ move in a symmetric manner relative to the center of the regular m-gon, and thus at time 1 they again form a regular m-gon with the same center. The same applies to all n/m groups, and since the robots are synchronous, at time 1 they together form a collection of n/m regular m-gons all having the same center. Since the robots in the same group have observed the same robot distributions, their next move at time 1 are also symmetric relative to the center of the regular m-gon they currently form. So again, at time 2 the robots form a collection of n/m regular m-gons all having the same center. Continuing in the same manner, we can prove that at any time instant t, the robots form a collection of n/m regular m-gons all having the same center. □

Conversely, we have:

Lemma 5 *For any multiset F of points that can be partitioned into n/m regular m-gons all having the same center, there exists a deterministic algorithm ψ for forming a figure similar to F starting from the initial configuration. (The algorithm does not depend on the initial configuration.)*

Proof We fix a total ordering over views. Also, we fix an ordering of the n/m regular m-gons in F. The idea is to move the robots in the j-th group in the ordering of views to the corners of the j-th regular m-gon. Specifically, first the robots execute Algorithm $\psi_{getview}$, so each robot knows the rank of the group it belongs to. (Note that the robots can detect the termination of $\psi_{getview}$ simultaneously.) The robots in the first group need not move any more, since the m-gon they form are similar to the corners of the 1st m-gon of F (except when the 1st m-gon is a point, in which case the robots must move to the point). Then each robot in the 2nd group computes the position of a corner of the 2nd m-gon of F (relative to the location of the 1st m-gon of F) that is closest to its current position (breaking ties in any deterministic manner), and moves to that position. The robots in the 3rd group continues next, and so on. Then eventually, a figure similar to F is formed. □

The following theorem summarizes the discussion given above.

Theorem 11 *Let m be the size of a largest subset of robots having an equivalent view at time 0. There exists a deterministic algorithm ψ for forming a figure similar to a multiset F of points iff F can be partitioned into n/m regular m-gons all having the same center.*

Proof The theorem follows from Lemmas 4 and 5. □

10 Concluding Remarks

We viewed a group of mobile robots as a distributed system in which the components can communicate with each other only by means of their moves, and investigated the possibility and impossibility of solving some of the problems related to the formation of geometric figures in the plane. Our study indicates that the assumptions we make on the knowledge and capabilities of the robots can affect the difficulty of solving the given problem in a subtle way. We are currently conducting similar investigations on (1) randomized algorithms, (2) the case in which the motion of a robot is not instantaneous, and (3) the 3-dimensional case. The results will be reported in a future paper.

Appendix A

Under the assumption that clones do not exist, the following algorithm $\psi_{scatter}$ transforms any given distribution of any number of robots (in which some robots may occupy the same position) into one in which (1) no two robots occupy the same position, and (2) the robots are not located on a single line segment if $n > 2$. Part 1 of $\psi_{scatter}$ moves all the robots to distinct positions, and Part 2 assures that the robots do not form a single line segment. The idea in Part 1 is the following. Suppose r_i and r_j occupy the same position initially. We let each robot move repeatedly, in the

directions of their respective positive x-axes, over distances that depend on the unit distances of their respective local coordinate systems. Then since r_i and r_j are not clones, either they move in different directions, they move over different distances, or eventually only one of them becomes active and move. So r_i and r_j eventually distinct positions. We do this in such a way that no two robots that occupy distinct positions initially will move to the same position.

Algorithm $\psi_{scatter}$
Part 1: Suppose that robot r_i becomes active and finds that not all robots occupy distinct positions. (Otherwise, Part 1 is over.) Let P_r be the positions of the robots that r_i observes, given in terms of Z_r.

Case 1.1 If no other robot is located at the current position of r_i, then r_i does not move, until it (becomes active later and) observes that all robots occupy distinct positions.

Case 1.2 On the other hand, if m other robots are located at the current position of r_i, where $m \geq 1$, then r_i finds the distance $a_i > 0$ to its nearest neighbor (excluding those at the current position of r_i), and computes the value $b_i = a_i f(a_i)/2$, where for $x \geq 0$, $f(x) = 1 - 1/2^x$ is a monotonically increasing function with range $[0, 1)$. (Note that a_i is given in terms of Z_i. So if another robot r_j located at the same position as r_i finds its own a_j in terms of Z_j, then $f(a_i) \neq f(a_j)$ unless the unit distances of Z_i and Z_j are identical.) Then r_i moves over distance $b_i/2$ in the positive x-direction of Z_i. After that, each time r_i becomes active and finds that there are still m other robots at the same position as itself, r_i moves over distance $b_i/2^k$ in the same direction, if that is the k-th move ($k = 2, 3, \ldots$). When r_i eventually becomes active and finds that there are fewer than m other robots at the same position as itself, r_i repeats this entire procedure from the beginning of **Case 1.2**, using a new value of a_i (and resetting k). This process is repeated each time r_i finds that fewer robots are located at the same position as itself, until no other robot is found to occupy the same position as r_i. Then r_i waits, without moving, until all robots occupy distinct positions.

Then r_i proceeds to Part 2.
Part 2: (At this moment, all the robots occupy distinct positions.) Suppose that robot r_i becomes active and finds that the robots are located on a single line segment. (Otherwise Part 2 is over.) If r_i is located at an endpoint of the segment, then it does not move. If r_i is not at an endpoint, then r_i moves over any distance in the direction perpendicular to the segment. (As soon as some robot does this, the robots no longer form a single line segment, and of course, all the robots occupy distinct positions.) □

Lemma 6 *Algorithm $\psi_{scatter}$ transforms any given distribution of the robots into one in which (1) no two robots occupy the same position and (2) the robots are not located on a single line segment if $n > 2$.*

Proof In Part 1, no two robots that are at distinct positions ever occupy the same position, since during the execution of **Case 1.2** for each fixed value of m, robot r_i moves over distance at most

$$b_i/2 + b_i/4 + \cdots < b_i = a_i f(a_i)/2 < a_i/2,$$

where a_i is the distance from r_i to any nearest robot not located at the same position at the beginning of **Case 1.2**. Since clones do not exist, any robot r_j that initially occupies the same position as r_i eventually moves to a different position than r_i, since either (a) their x-axes have different orientations, (b) their unit distances are different and thus $f(a_i) \neq f(a_j)$, or (c) only one of them becomes active and moves. In Part 2, since the robots at the endpoints of the line segment do not move, as soon as any one robot moves as specified, that robot and the robots at the endpoints form a triangle. □

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Fig. 2

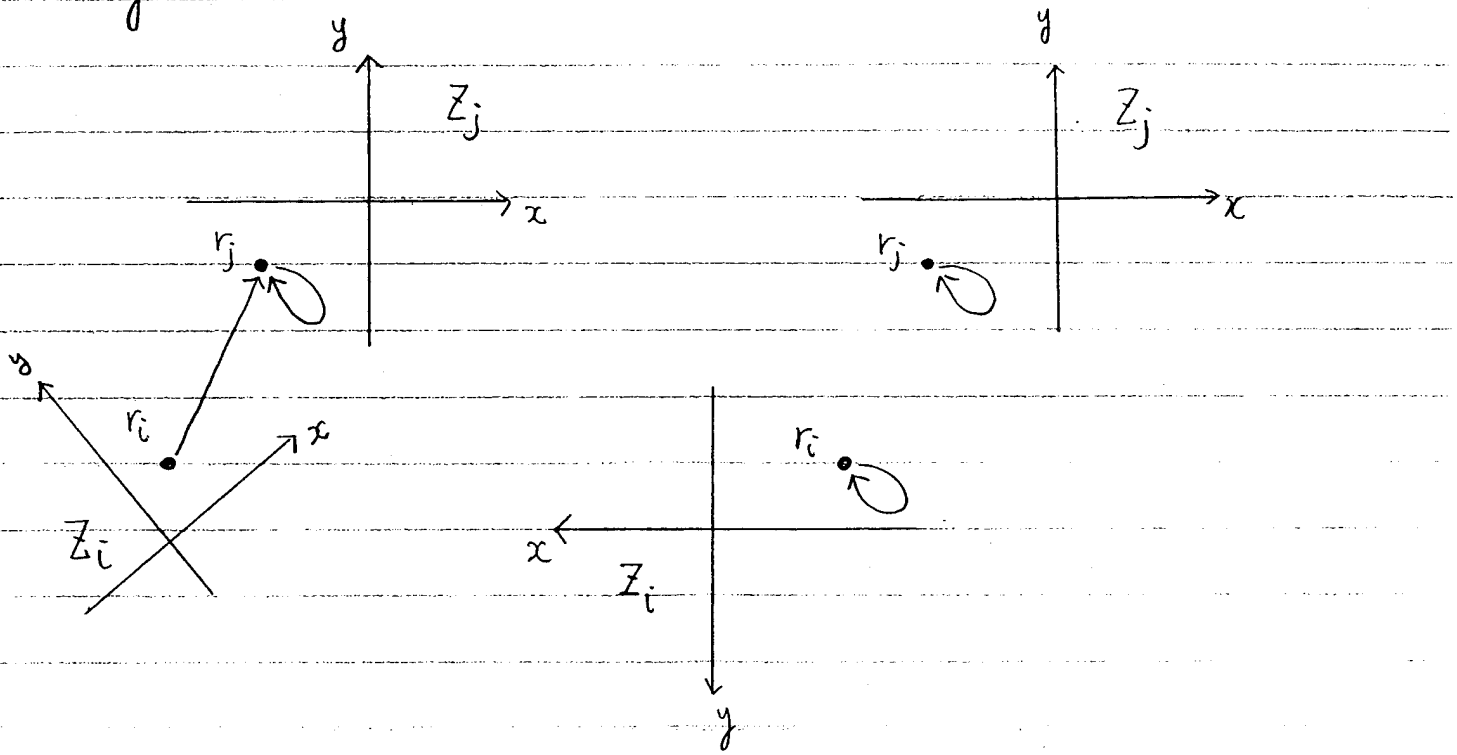

Fig. 3

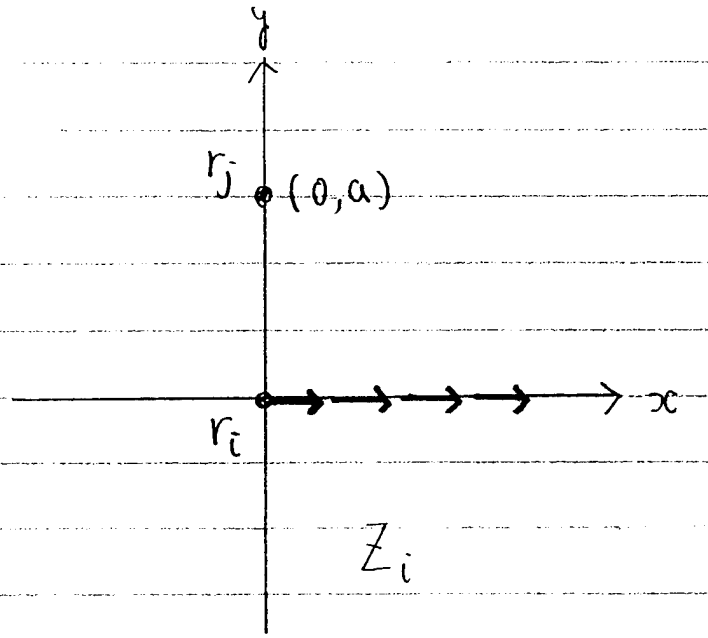

Fig. 4

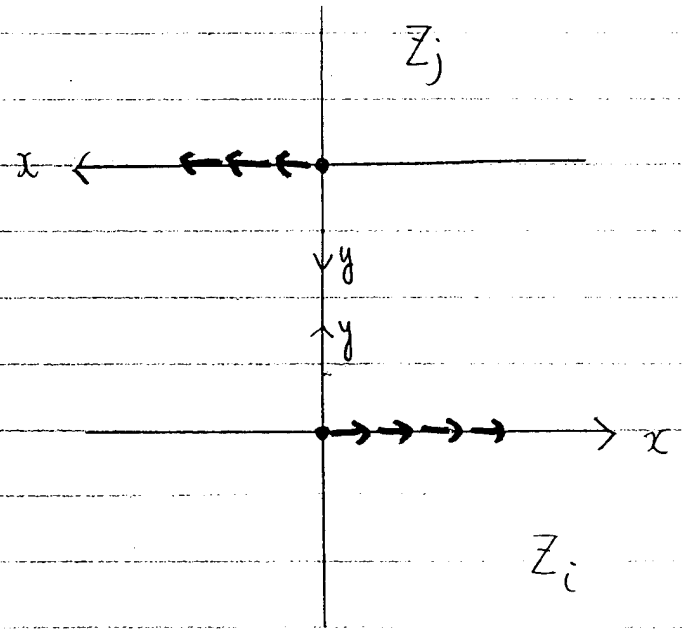

Fig. 5

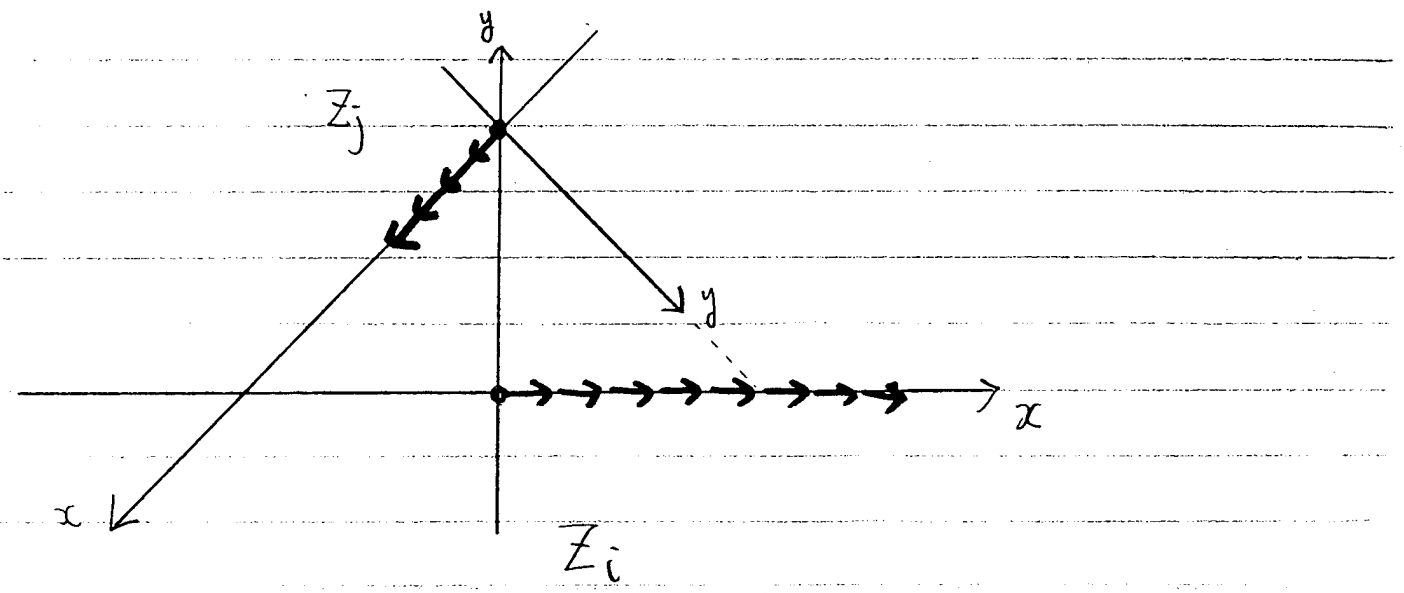

Fig. 6

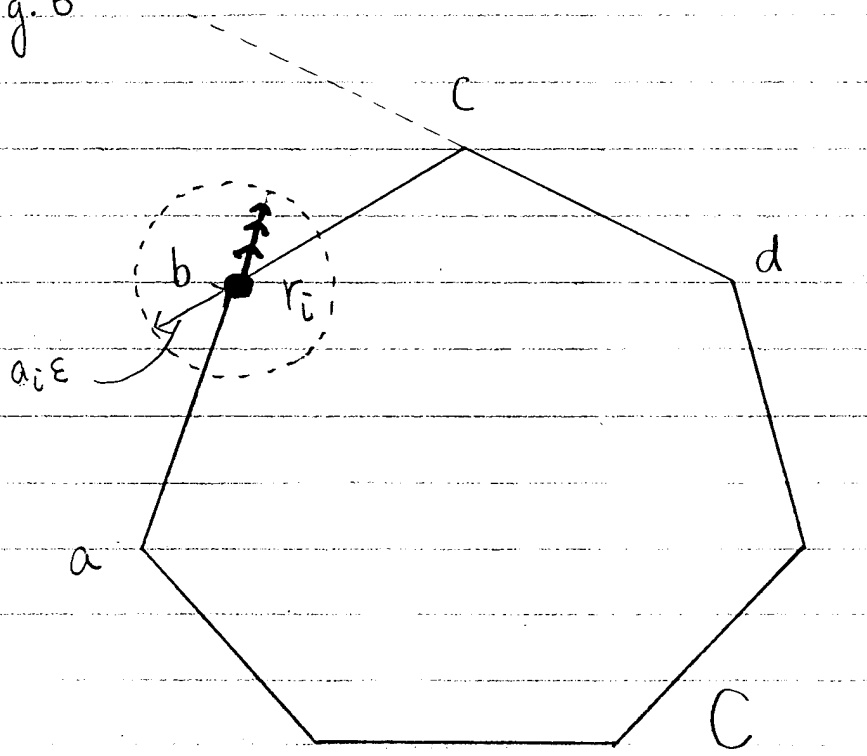

Fig. 7

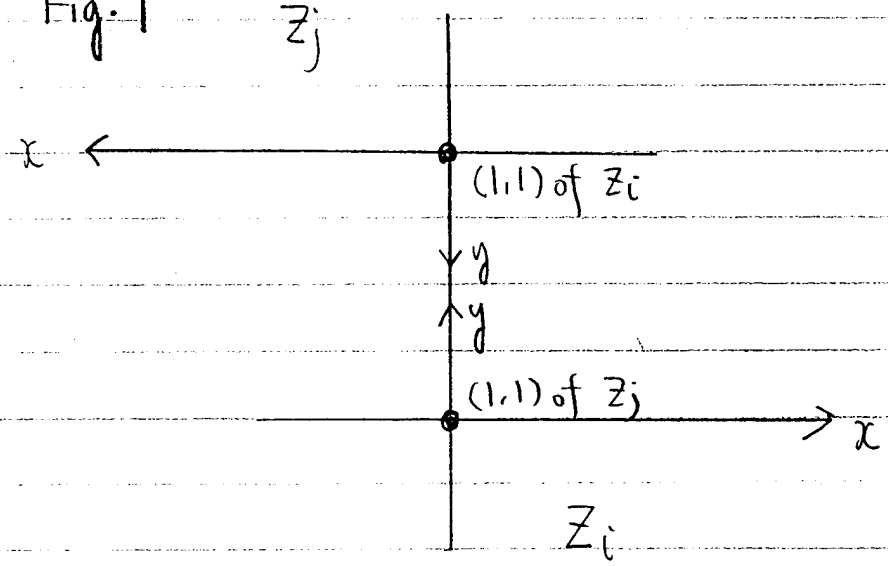

Fig. 8

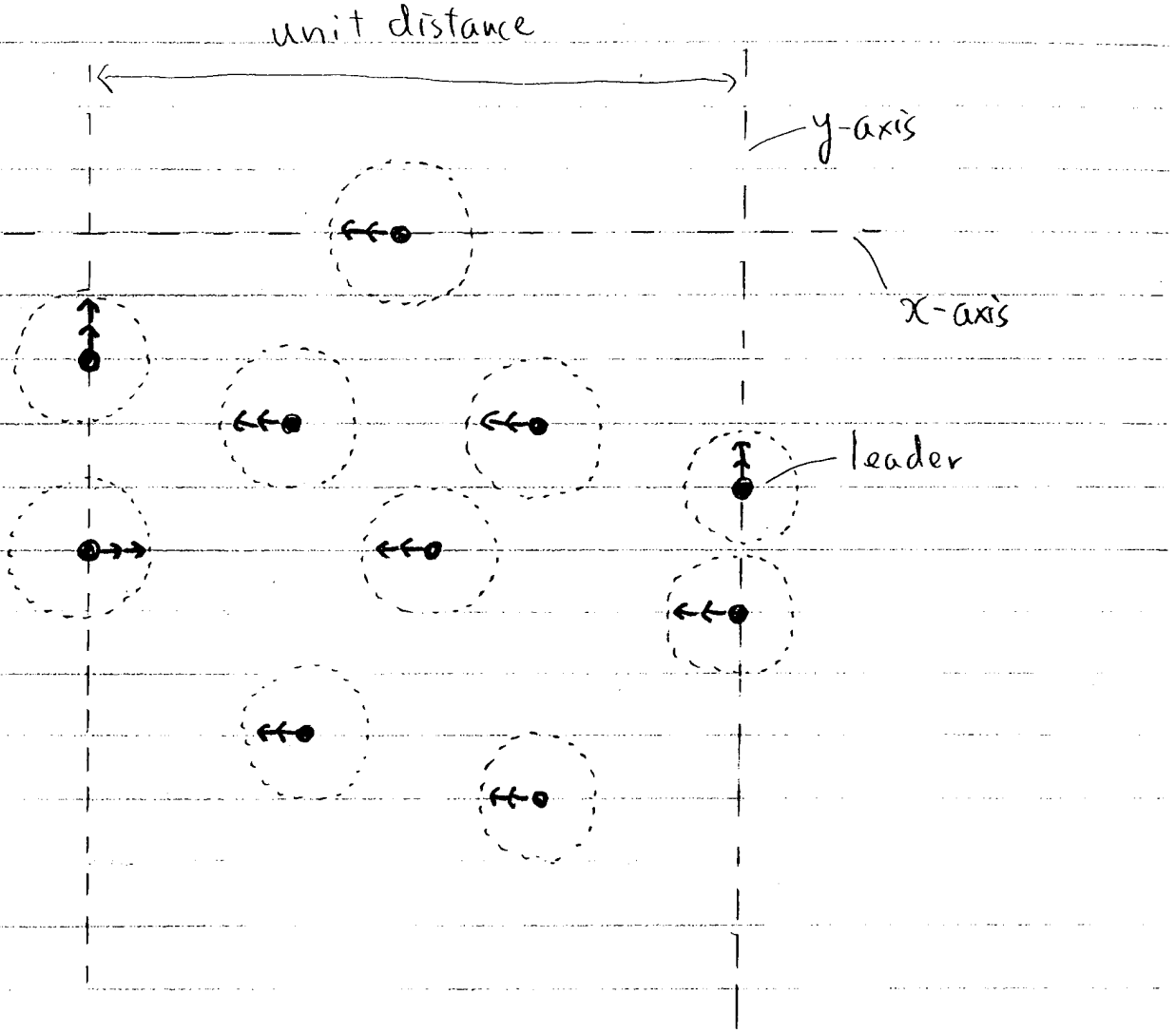

Fig. 9

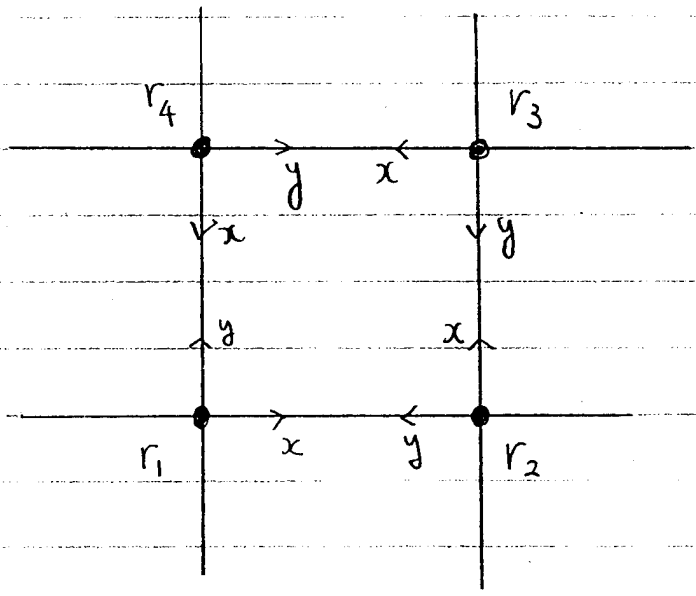

Fig. 10

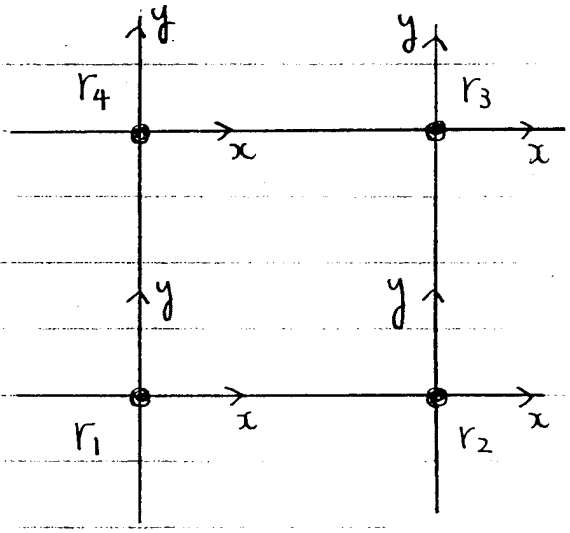

Fig. 11